Analytical Control of SISO Nonlinear Processes with Input Constraints

Sairam Valluri and Masoud Soroush
Dept. of Chemical Engineering, Drexel University, Philadelphia, PA 19104

A new method of controlling nonlinear processes with actuator saturation nonlinearities is presented. Two nonlinear control laws are derived for single-input, single-output nonlinear processes. Whether input constraints are present or not, each of these dynamic control laws can minimize the mismatch between the closed-loop output response and the nominal linear output response that the same control law induces when there are no constraints. The control laws offer great flexibility to obtain a desirable closed-loop output response in the presence of active input constraints, and they inherently compensate for windup. They allow one to adjust the time period during which an input constraint is active and the decay rate of the mismatch between the constrained output response and the nominal linear unconstrained output response. Conditions under which the constrained closed-loop system is asymptotically stable are given. The connections between the developed control laws and the modified internal model control are established. The performance of the control laws is demonstrated by numerical simulations of chemical and biochemical reactor examples.

Introduction

In every physically meaningful process, process variables are naturally bounded. For example, the magnitude of a manipulated variable can only vary between the upper and lower limits of the corresponding actuator. Ignoring the boundedness of manipulated variables at the stage of controller design may lead to significant deterioration in the closed-loop performance and even closed-loop instability.

In the case of model predictive control (MPC), the constraints are explicitly accounted for, and the controller action is solution to a constrained optimization problem. Furthermore, tunable parameters such as prediction and control horizons can be adjusted to achieve a desirable closed-loop response with guaranteed closed-loop stability in the presence of constraints (Tsang and Clarke, 1988; Eaton and Rawlings, 1992; Michalska and Mayne, 1993; Meadows et al., 1995). However, in the case of analytical (non-MPC) control, a controller is usually designed assuming that there is no bound, and then the controller is properly modified to prevent significant performance deterioration due to the bounds.

A major deterioration in the performance of an analytical controller is windup, which occurs when (a) the output of a dynamic controller, especially with a slow or unstable mode, exceeds the actuator limit, and (b) the state variables of the controller are not aware of the saturation and thus are wrongly updated. In linear analytical control, the issues of windup and constraint handling as well as closed-loop stability in the presence of input constraints have been studied extensively (Åström and Rundqwist, 1989; Walgama and Sternby, 1990; Kothare et al., 1994; Kapoor and Daoutidis, 1997). In particular, in linear analytical model-based control, powerful results are available in the frameworks of internal model control (Zheng et al., 1994) and model state feedback control (Coulibaly et al., 1995).

Differential geometric control methods fall in the class of nonlinear analytical model-based control. Because feedback linearization is only possible in the absence of constraints, much of the pioneering work on differential geometric control has been focused on unconstrained processes. In recent years, there has been a growing research activity in the area of differential geometric control to address the practically-important problems of windup and input constraints. In particular, several approaches to the problem of integral windup in input-output linearizing control have been proposed. These include conditional integration (Soroush and Kravaris, 1992a),

Correspondence concerning this article should be addressed to M. Soroush.

constraint mapping of Calvet and Arkun (1988) together with linear MPC (Oliveira et al., 1995; Kurtz and Henson, 1996), the constraint mapping together with the modified linear internal model control (Doyle, 1995), an anti-windup observer-based scheme for nonlinear processes (Kapoor and Daoutidis, 1996) and MPC formulation of input-output linearization (Soroush and Kravaris, 1996; Valluri et al., 1997). The MPC formulation of input-output linearization allows one to address *directly* the problems of input constraint handling and windup in input-output linearizing control.

In this article, a new method for control of single-input single-output processes with input constraints is presented. Here are given two nonlinear control laws that are input-output linearizing in the absence of input constraints and that can provide significant improvement in the closed-loop response in the presense of active input constraints. The control laws can be tuned such that the mismatch between the nominal linear unconstrained closed-loop response and the constrained closed-loop response is minimized over a very short time horizon into the future. The connections between the derived control laws and (a) those given in (Valluri et al., 1997), (b) the observer-based approach of Kapoor and Daoutidis (1996), the modified internal model control-based approach of Doyle (1995), (c) the modified internal model control (IMC) (Zheng et al., 1994), and (d) the model state feedback control (MSFC) (Coulibaly et al., 1995) are established.

This article begins with a description of the scope of the work. As a motivation, a simple chemical reactor example is considered in the Motivation and Problem Statement section followed by an overview of the solution. Dynamic control laws are derived first for processes with full state measurements and then for processes with incomplete state measurements and deadtime in the following section. Two theorems on the closed-loop stability and some guidelines for tuning the controllers are given. The derived control laws are applied to linear systems so as to establish their connections with the modified linear IMC and MSFC, and the application and performance of the nonlinear control laws are demonstrated by chemical and biochemical reactor examples.

Scope

We consider the class of single-input single-output, continuous-time, nonlinear processes described by a state-space model of the form

$$\begin{cases} \dot{x}(t) = f[x(t)] + g[x(t)]u(t), \quad x(0) = 0 \\ y(t) = h[x(t-\theta)] \end{cases} \tag{1}$$

where θ is the deadtime (s), $x = [x_1 \ \cdots \ x_n]^T$ is the vector of state variables, u is the manipulated input, and y is the controlled output. We make the following assumptions: (a1) every variable of the system of Eq. 1 is in the form of deviation from its nominal steady-state value, and thus the origin is the nominal equilibrium point; (a2) $x \in X \subset \mathbb{R}^n$, where X is an open connected set which contains the equilibrium point; (a3) $u \in U = \{u | u_{min} \leq u \leq u_{max}\} \subset \mathbb{R}$, where u_{min} and u_{max} are scalar constants which satisfy $u_{min} < 0 < u_{max}$; (a4) $f(x)$ and $g(x)$ are smooth vector functions on X; (a5) $h(x)$ is a smooth scalar function on X; (*a*6) output set point, denoted by y_{sp}, is *achievable* in the sense that there exists a $u_o \in \text{int}(U)$, which satisfies $f(\zeta) + g(\zeta)u_0 = 0$, where $\zeta \in X$ and $h(\zeta) = y_{sp}$; (a7) $L_g L_f^{r-1} h(x) \neq 0$, $\forall x \in X$, where r is the relative order of the delay-free output y_o, $[y_o(t) = h(x(t))]$, [recall that r is the smallest integer for which $L_g L_f^{r-1} h(x) \not\equiv 0$]; and (a8) the delay-free part of process (system of Eq. 1 with $\theta = 0$) is minimum phase (has asymptotically stable zero dynamics) on X.

The measured (actual) value of the controlled variable, denoted by $\bar{y}$, is given by

$$\bar{y}(t) = h[\bar{x}(t-\theta)] + d(t)$$

where $\bar{x}$ represents the measured (actual) value of the vector of state variables and d denotes an unmeasurable *constant* output disturbance. The model-predicted values of the controlled variable and the vector of state variables will be denoted by y and x, respectively. The following saturation function will be used throughout this article

$$\text{sat}\{w\} \overset{\Delta}{=} \begin{cases} u_{min}, & \text{if } w < u_{min} \\ w, & \text{if } u_{min} \leq w \leq u_{max} \\ u_{max}, & \text{if } w > u_{max} \end{cases} \tag{2}$$

Motivation and Problem Statement

Most of nonlinear analytical controllers are input-output linearizing. They induce linear input-output closed-loop response in the absence of constraints and possess integral action to ensure an offsetless closed-loop response in the presence of constant disturbances and model errors. If in the synthesis of these controllers, the boundedness of manipulated inputs is ignored, the closed-loop performance may be significantly poorer than the integral of squared error (ISE) optimal response that can be achieved in the presence of the same input constraints.

To illustrate some of the possible detrimental effects of input constraints on the closed-loop performance of such a controller, the closed-loop response of a chemical reactor example under an input-output linearizing controller is simulated. The same example is used later to show the significantly "better" response that one of the controllers presented in this article can induce in the presence of the same input constraints.

Motivation

Consider the same continuous stirred tank reactor (CSTR) example described in (Soroush and Kravaris, 1992b). The reactor is represented by

$$\begin{cases} \dfrac{d\bar{C}_A}{dt} = R_A(\bar{C}_A, \bar{T}) + \dfrac{C_{A_i} - \bar{C}_A}{\tau} \\ \dfrac{d\bar{T}}{dt} = \dfrac{R_H(\bar{C}_A, \bar{T})}{\rho c} + \dfrac{T_i - \bar{T}}{\tau} + \dfrac{Q}{\rho c V} \end{cases} \tag{3}$$

Let us assume that both state variables are measurable and that the reactor temperature $\bar{T}$ is to be maintained at 400 K ($y_{sp} = 400$) by manipulating the heat input to the reactor Q (kJ·s^{-1}).

The process model of Eq. 3 can be recast in form of Eq. 1 with $\theta = 0$: $\bar{y} = \bar{T}$, $\bar{x} = [\bar{C}_A\ \bar{T}]^T$, $u = Q$. For this process, $r = 1$ ($L_g h = (1/\rho c V) \neq 0$). If there are no active constraints, the specified control objective can be accomplished in a straightforward manner by means of a mixed error- and state-feedback globally linearizing controller (Daoutidis and Kravaris, 1992)

$$\begin{cases} \dfrac{d\eta_1}{dt} = \dfrac{1}{\gamma_1} e, & \eta_1(0) = \bar{T}(0) \\ w = \dfrac{\rho c V}{\gamma_1}\left(\eta_1 + e - \bar{T} - \gamma_1\left[\dfrac{R_H(\bar{C}_A, \bar{T})}{\rho c} + \dfrac{T_i - \bar{T}}{\tau}\right]\right) \end{cases} \tag{4}$$

where $e = y_{sp} - \bar{y}$. The preceding dynamic controller has integral action, which ensures a satisfactory regulatory performance and robustness.

If there is no limit on the rate of heat input to the reactor Q, then $Q = w$, and the input-output closed-loop response will be exactly the requested, linear, first-order response $100\, d\bar{T}/dt + \bar{T} = T_{sp}$. Suppose the rate of heat input is limited: $|Q| \leq Q_{\max}$. In this case, if the rate of heat input calculated by the controller (w) is, for example, higher than $Q_{\max}$, then the actual rate of heat input $Q = Q_{\max} \neq w$ leading to a mismatch between the controller output and the reactor input.

Figure 1 depicts the profiles of the reactor temperature and heat input rate under the nonlinear controller of Eq. 4 with $\gamma_1 = 100$ s. The solid line (Case A1) represents the response when there is no limit on the rate of heat input (that is, $|Q| \leq \infty$), and Cases A2, A3, and A4 correspond to $|Q| \leq 15$ $\text{kJ}\cdot\text{s}^{-1}$, $|Q| \leq 10$ $\text{kJ}\cdot\text{s}^{-1}$ and $|Q| \leq 4$ $\text{kJ}\cdot\text{s}^{-1}$, respectively. As can be seen in this figure, the temperature response in Cases A2, A3 and A4 exhibits significant overshoot and longer settling time, compared to the unconstrained response.

Tighter limits on the heat input rate (smaller values of $Q_{\max}$) lead to a longer settling time, but the same trend does not occur with regard to overshoot. If a much faster nominal closed-loop response is requested by choosing a smaller value for γ_1, the closed-loop response will exhibit stable limit cycle.

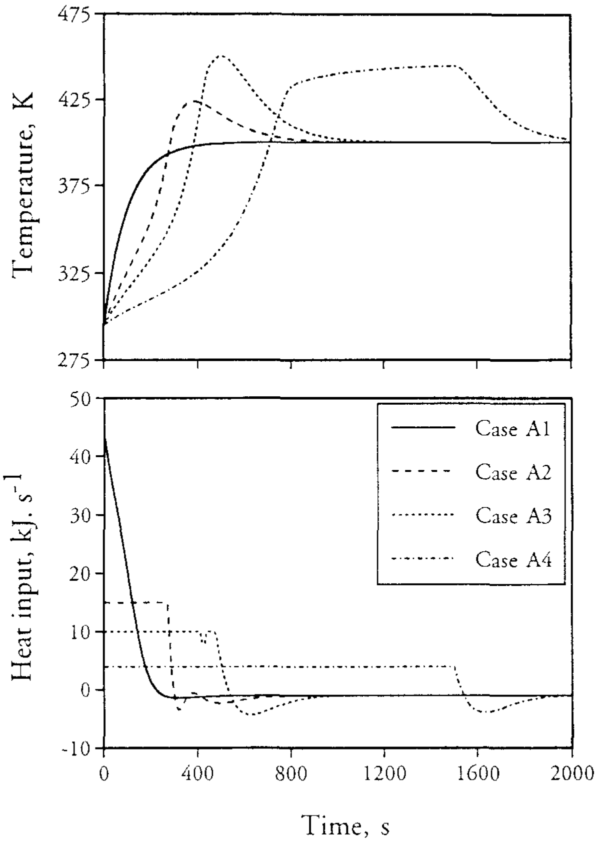

Figure 1. Startup profiles of the controlled and manipulated variables of the chemical reactor example under the controller of Eq. 4.

A1, $|Q| < \infty$; A2, $|Q| \leq 15$ $\text{kJ}\cdot\text{s}^{-1}$; A3, $|Q| \leq 10$ $\text{kJ}\cdot\text{s}^{-1}$; A4, $|Q| \leq 4$ $\text{kJ}\cdot\text{s}^{-1}$.

Problem statement and overview of the solution

As shown in the previous subsection, the closed-loop performance under a constrained input-output linearizing controller with integral action may be significantly poorer than an ISE optimal response that can be achieved in the presence of the same input constraints. To derive control laws that in the presence of input constraints can induce a closed-loop response with a significantly lower ISE and even with minimum ISE, we limit our focus to the nonlinear control laws that: (a) in the absence of input constraints induce an adjustable linear input-output closed-loop response; (b) in the presense of input constraints force the constrained output response to converge to the linear unconstrained input-output closed-loop response at an adjustable rate; and (c) allow one to adjust the time period during which the plant input is in saturation.

In mathematical terms, we seek analytical nonlinear control laws that:

(a) in the absence of input constraints induce a linear input-output closed-loop response of the form of

$$\bar{y}(t) + \gamma_1 \frac{d\bar{y}(t)}{dt} + \cdots + \gamma_r \frac{d^r \bar{y}(t)}{dt^r} = y_{sp}(t - \theta) \tag{5}$$

where $\gamma_1, \ldots, \gamma_r$ are scalar adjustable parameters with $\gamma_r \neq 0$;
(b) can minimize the quadratic performance index

$$\|\hat{y}(\tau) - y^*(\tau)\|_p^2 \tag{6}$$

subject to the constraints

$$u_{\min} \leq u(t) \leq u_{\max}$$

where t represents the present time (s), $\|\omega(\tau)\|_p$ denotes the p-function norm of the scalar function $\omega(\tau)$ over the finite time interval $[t + \theta, t + \theta + T_h]$, $y^*(\tau)$ is the desirable, linear, closed-loop, output response that is obtained in the absence of the input constraints (that is, that of Eq. 5), $\hat{y}(\tau)$ is the predicted value of the controlled output when the constraints are present, and T_h is a sufficiently short time horizon into the future.

(c) when the input constraints are not active, eliminate the mismatch between $\tilde{y}$ and y^* at an adjustable decay rate. In particular, we request the mismatch to be governed by

$$[\tilde{y}(t)-y^*(t)]+\beta_1\frac{d[\tilde{y}(t)-y^*(t)]}{dt}+\cdots+\beta_r\frac{d^r[\tilde{y}(t)-y^*(t)]}{dt^r}=0 \quad (7)$$

where $\beta_1, \ldots, \beta_r$ are scalar adjustable parameters with $\beta_r \neq 0$. This allows one to adjust the rate at which the constrained response approaches the unconstrained linear input-output response, without changing the nominal linear unconstrained output response of Eq. 5.

To derive control laws with the aforementioned theoretical properties, we solve the following constrained minimization problem

$$\min_{u(t)} \|\hat{y}(\tau)-\tilde{y}^*(\tau)\|_p^2 \quad (8)$$

subject to the constraints $u_{\min} \leq u(t) \leq u_{\max}$. Here $\tilde{y}^*(t)$ describes the path that the output response will follow to reach the unconstrained response y^* when the constraints are no longer active. It is the solution to the linear system

$$\tilde{y}^*(\tau)+\beta_1\frac{d\tilde{y}^*(\tau)}{d\tau}+\cdots+\beta_r\frac{d^r\tilde{y}^*(\tau)}{d\tau^r}=\tilde{y}(t) \quad (9)$$

subject to the initial conditions

$$\tilde{y}^*(t+\theta)=h(x(t))+\tilde{y}(t)-h(x(t-\theta))$$

$$\frac{d^l\tilde{y}^*(t+\theta)}{dt^l}=L_f^l h(x(t)), \qquad l=1,\ldots,r-1$$

where the forcing function $\tilde{y}$ is related to the output setpoint y_{sp} according to

$$\begin{cases} y^*(t+\theta)+\sum_{l=1}^{r}\gamma_l\frac{d^l y^*(t+\theta)}{dt^l}=y_{sp}(t) \\ \tilde{y}(t)=y^*(t+\theta)+\sum_{l=1}^{r}\beta_l\frac{d^l y^*(t+\theta)}{dt^l} \end{cases} \quad (10)$$

In the limit that $\beta_1, \ldots, \beta_r \to 0$, according to Eqs. 9 and 10, $\tilde{y}^* \to y^*$. Thus, in that limit the solution to the constrained minimization problem of Eq. 8 will also be the solution to the constrained minimization problem of Eq. 6. Using rth-order truncated Taylor series expansions of $\hat{y}(\tau)$ and $\tilde{y}^*(\tau)$ about $(t+\theta)$, as in (Valluri et al., 1997), we see that the solution to the constrained minimization problem of Eq. 8 is given by the state feedback

$$u(t)=\mathrm{sat}\left\{\frac{\tilde{y}(t)-\bar{y}(t)+h(x(t-\theta))-h(x(t))-\sum_{l=1}^{r}\beta_l L_f^l h(x(t))}{\beta_r L_g L_f^{r-1}h(x(t))}\right\}.$$

The following notations will be used in the subsequent parts of this article:

- *The linear time-invariant system*

$$\begin{cases} \dot{\eta}=A_c\eta+b_c\tilde{y} \\ \tilde{y}^*=c_c\eta \end{cases} \quad (11)$$

where $\eta \in \mathbb{R}^{r\times 1}$, and A_c, b_c and c_c are matrices of appropriate dimensions (representative matrices are given in the Appendix A), will denote a minimal-order state-space realization of the linear system of Eq. 9.

- *The linear time-invariant system*

$$\begin{cases} \dot{\xi}=A_c^*\xi+b_c^* y_{sp} \\ \tilde{y}=\frac{\beta_r}{\gamma_r}c_c^*\xi+\frac{\beta_r}{\gamma_r}y_{sp} \end{cases} \quad (12)$$

where $\xi \in \mathbb{R}^{r\times 1}$, and A_c^*, b_c^*, and c_c^* are matrices of appropriate dimensions (representative matrices are given in the Appendix A), will represent a minimal-order state-space realization of the linear system of Eq. 10.

Nonlinear Controllers Synthesis

This section includes two theorems that concisely describe the theoretical properties of two nonlinear control laws. These properties include their ability to minimize the performance index of Eq. 6, to handle input constraints effectively, and to eliminate the effect of unmeasurable constant disturbances and model errors asymptotically. Nonlinear processes with full state measurements are first considered, followed by nonlinear processes with incomplete state measurements and deadtime.

Processes with full state measurements

Theorem 1. For a process of the form of Eq. 1 with complete state measurements ($\bar{x}$) and no deadtimes, the dynamic mixed error- and state-feedback control law

$$\begin{cases} \dot{\eta}=A_c\eta+b_c\Phi(\bar{x},u), & \eta(0)=0 \\ \dot{\xi}=A_c^*\xi+b_c^*[c_c\eta+e], & \xi(0)=0 \\ u=\mathrm{sat}\{\Psi_0(\bar{x},c_c\eta+e+c_c^*\xi)\} \end{cases} \quad (13)$$

where

$$\Phi(\bar{x},u)=h(\bar{x})+\sum_{l=1}^{r}\beta_l L_f^l h(\bar{x})+\beta_r L_g L_f^{r-1}h(\bar{x})u,$$

$$\Psi_0(\bar{x},\zeta)\triangleq\frac{\zeta-\frac{\gamma_r}{\beta_r}h(\bar{x})-\sum_{l=1}^{r}\gamma_r\frac{\beta_l}{\beta_r}L_f^l h(\bar{x})}{\gamma_r L_g L_f^{r-1}h(\bar{x})},$$

(a) minimizes the performance index of Eq. 6 in the limit that all the roots of

$$\beta_r s^r+\cdots+\beta_1 s+1=0 \quad (14)$$

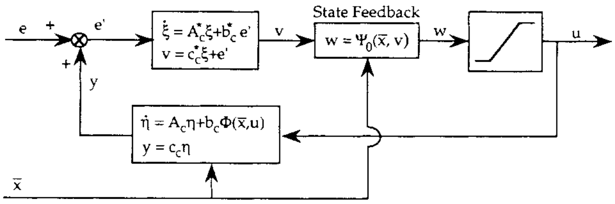

Figure 2. Generalized, mixed error- and state-feedback control structure.

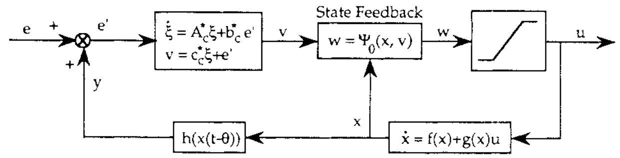

Figure 3. Generalized, reduced-order, error-feedback control structure.

are placed far left in the complex plane (that is, in the limit that $\beta_l \to 0$, $l = 1, \ldots, r$).

(b) in the absence of the constraints, induces the linear input-output closed-loop response of Eq. 5 with $\theta = 0$.

(c) has integral action: in the presence of constant disturbances and model errors, induces an offsetless closed-loop response.

The proof is given in the Appendix B.

The block diagram of the control law of Theorem 1 is depicted in Figure 2. This control structure will be referred to as the generalized, mixed error- and state-feedback control structure. The ξ subsystem provides extra flexibility for achieving a desirable response in the presense of input constraints.

The theoretical connections between the control law of Eq. 13 and those of Doyle (1995), Kapoor and Daoutidis (1996) and (Valluri et al., 1997) are as follows:

- The mixed error- and state-feedback control law of Eq. 13 is a minimal-order state-space realization of the control law of Doyle (1995) which has r redundant modes. The elimination of these redundant modes leads to the control law of Eq. 13. Because of its mixed error- and state-feedback nature, the application of the control law of Doyle to linear time-invariant processes does not lead to a modified internal model controller.
- In the special case that the tunable parameters are chosen such that $\beta_l = \gamma_l$, $l = 1, \ldots, r$, the vector $c_c^* = [0 \cdots 0]$, and $\beta_l \gamma_r / \beta_r = \gamma_l$, $l = 1, \ldots, r$. In such a case, the control law of Eq. 13 will be of order r and will be exactly that of Theorem 3 in (Valluri et al., 1997), which was derived using a different nonlinear MPC approach. This rth-order controller belongs to the class of mixed error- and state-feedback controllers developed by Kapoor and Daoutidis (1996) by using an observer-based approach to windup compensation in nonlinear processes.

Processes with incomplete state measurements and deadtime

Theorem 2. For a process of the form of Eq. 1 with incomplete state measurements, the dynamic error-feedback control law

$$\begin{cases} \dot{\xi} = A_c^* \xi + b_c^* e', & \xi(0) = 0 \\ \dot{x} = f(x) + g(x)\text{sat}\{\Psi_0(x, e' + c_c^* \xi)\}, & x(0) = 0 \\ u = \text{sat}\{\Psi_0(x, e' + c_c^* \xi)\} \end{cases} \quad (15)$$

where

$$e'(t) = e(t) + h[x(t - \theta)],$$

(a) minimizes the performance index of Eq. 6 in the limit that all the roots of Eq. 14 are placed far left in the complex plane (that is, in the limit that $\beta_l \to 0$, $l = 1, \ldots, r$).

(b) in the absence of the constraints, induces the linear input-output closed-loop response of Eq. 5.

(c) has integral action: in the presence of constant disturbances and model errors, induces an offsetless closed-loop response.

The proof is given in the Appendix B.

The block diagram of the control law of Theorem 2 is depicted in Figure 3. This control structure will be referred to as the generalized, reduced-order, error-feedback control structure. Again, here, the ξ subsystem provides extra flexibility for obtaining a better closed-loop performance in the presence of input constraints. Choosing the controller tunable parameters according to $\beta_l = \gamma_l$, $l = 1, \ldots, r$ leads to the elimination of the effect of ξ subsystem on the controller action. In such a case, the control law of Eq. 15 will be of order n and will be exactly the reduced-order, error-feedback control law given in (Valluri et al., 1997).

Modified IMC parameterization of the control law of Eq. 15

The error-feedback control law of Eq. 15 can be parameterized according to the modified linear IMC structure (Zheng et al., 1994), leading to the controller components given in Table 1, where $u = \text{sat}\{w_1 - w_2\}$. Unlike in the modified linear IMC structure, in this nonlinear case, Q_1 depends on the state x (and thus on u). For this reason, this control structure will be referred to as the nonlinear modified IMC structure, whose block diagram is shown in Figure 4. The above parameterization indicates that the implementation of the control law of Eq. 15 according to the modified IMC structure will increase the order of the controller by $2n$. Thus, the control law of Eq. 15 is a minimal-order state-space realization of a nonlinear modified IMC controller. As we will see, when the control law of Eq. 15 is applied to linear time-invariant processes, the resulting linear control law will be exactly (a) a minimal-order state-space realization of a modified IMC controller and (b) a model state feedback controller

Table 1. Modified IMC Parameterization of the Nonlinear Control Law of Eq. 15

P	Q_1	Q_2
$\dot{x} = f(x) + g(x)u$ $y(t) = h[x(t-\theta)]$	$\dot{x} = f(x) + g(x)u$ $\dot{\xi} = A_c^* \xi + b_c^* e'$ $w_1 = \dfrac{e' + c_c^* \xi}{\gamma_r L_g L_f^{r-1} h(x)}$	$\dot{x} = f(x) + g(x)u$ $w_2 = \dfrac{h(x) + \sum_{l=1}^{r} \beta_l L_f^l h(x)}{\beta_r L_g L_f^{r-1} h(x)}$

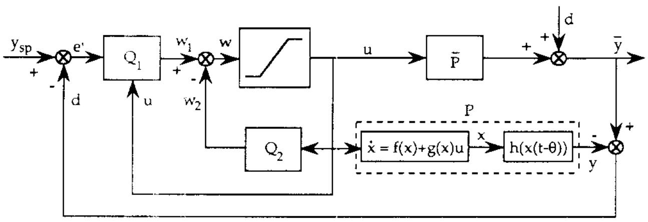

Figure 4. Nonlinear modified IMC structure.

when the tunable parameters are chosen such that $\beta_l = \gamma_l$, $l = 1, \ldots, r$.

Closed-Loop Stability and Guidelines for Tuning Controllers

Closed-loop stability

Consider the following conditions: (c1) the process of Eq. 1 with $\theta = 0$ is minimum-phase (has stable zero dynamics) on X; (c2) the process of Eq. 1 is asymptotically open-loop stable on X; (c3) the parameters $\gamma_1, \ldots, \gamma_r$ are chosen such that all the roots of the characteristic equation $\gamma_r s^r + \cdots + \gamma_1 s + 1 = 0$ lie in the left half of the complex plane, and (c4) the parameters $\beta_1, \ldots, \beta_r$ are chosen such that all the roots of the characteristic equation $\beta_r s^r + \cdots + \beta_1 s + 1 = 0$, lie in the left half of the complex plane.

Definition 1. For a process of the form of Eq. 1, an initial condition x_0 is said to be feasible, if and only if there exists a feasible smooth manipulated input trajectory $u(t)$ (that is, a smooth $u(t)$ that satisfies $u_{\min} \le u(t) \le u_{\max}$, $\forall t \in [0, t_f]$), that can drive the process from x_0 to the origin (nominal equilibrium point) within a finite time t_f. The set of all the feasible initial conditions will be denoted by χ.

Definition 2. A process of the form of Eq. 1 is said to be asymptotically stable on X, if and only if for every initial condition $x_0 \in X$, the system $\dot{x}(t) = f[x(t)]$ evolves such that $\lim_{t \to \infty} \|x(t)\| \to 0$.

Theorem 3. For a process of the form of Eq. 1 with complete state measurements ($\bar{x}$) and no deadtimes, the closed-loop system under the dynamic mixed error- and state-feedback control law of Eq. 13

(a) in the absence of input constraints will be asymptotically stable on X, if the conditions c1, c3 and c4 hold.

(b) in the presence of input constraints will be asymptotically stable on $X \cap \chi$, if the conditions c1, c3 and c4 hold and the tunable parameters $\beta_1, \ldots, \beta_r$ are chosen such that a root of Eq. 14 is placed sufficiently close to the origin.

The proof is given in the Appendix B.

Theorem 4. For a process of the form of Eq. 1 with incomplete state measurements, the closed-loop system under the dynamic error-feedback control law of Eq. 15:

(a) in the absence of input constraints will be asymptotically stable on X, if the conditions c1, c2, c3 and c4 hold.

(b) in the presence of input constraints will be asymptotically stable on $X \cap \chi$, if the conditions c1, c2, c3 and c4 hold and the tunable parameters $\beta_1, \ldots, \beta_r$ are chosen such that a root of Eq. 14 is placed sufficiently close to the origin.

The proof is given in the Appendix B.

Theorems 3 and 4 indicate that for a constrained nonlinear process of the form of Eq. 1, the derived control laws offer great flexibility to ensure closed-loop stability without changing the nominal closed-loop response that the same control laws can induce in the absence of constraints.

Guidelines for tuning the controllers

The tunable parameters $\gamma_1, \ldots, \gamma_r$ determine the shape of the nominal, linear, closed-loop response (described in Eq. 5) that is induced when there are no constraints. The tunable parameters $\beta_1, \ldots, \beta_r$ do not affect the shape of the nominal, linear, closed-loop response, but they influence the shape of the mismatch between the constrained closed-loop response and the nominal linear closed-loop response. They also affect the time period during which a manipulated input saturates; the further left the locations of the roots of the polynomial of Eq. 14, the longer the time period during which the manipulated input may stay at a limit. Indeed, when the manipulated input is not saturated, the control laws of Eqs. 13 and 15 both induce the same closed-loop response of Eq. 7; that is, the control laws of Eqs. 13 and 15 track y^* asymptotically when the manipulated input is not saturated.

The number of the controller tunable parameters can be decreased from $2r$ to 2 by placing (a) all the roots of the polynomial $\gamma_r s^r + \cdots + \gamma_1 s + 1 = 0$ at $s = -1/\epsilon$ and (b) all the roots of the polynomial $\beta_r s^r + \cdots + \beta_1 s + 1 = 0$ at $s = -1/\alpha$, where ϵ and α are positive scalar constants to be chosen. This can be achieved simply by setting

$$\gamma_l = \frac{r!}{(r-l)!l!}\epsilon^l, \quad \beta_l = \frac{r!}{(r-l)!l!}\alpha^l, \quad l = 1, \ldots, r$$

For low relative orders, these tunable parameter settings are given in Table 2. While a larger value of ϵ leads to a more sluggish nominal linear closed-loop response, a larger value of α results in less saturation of the manipulated input. With a smaller value of α, the control laws take the controlled output to its nominal linear unconstrained closed-loop response y^* faster. However, the controllers may not be able to maintain the controlled output at y^*, because this may require the controllers to take an infeasible aggressive action in the opposite direction. Thus, a response with a lower ISE may be obtained with increasing α (see the simulation results of the reactor example). It is noteworthy that in the presence of input constraints the minimizer of the performance index of Eq. 6 over a very short horizon may not be identical to the minimizer of the same performance index over an infinite horizon. However, in general, a long prediction-horizon, minimization problem does not have an analytical solution (does not lead to an analytical controller).

Application to Linear Systems

Consider the class of time-invariant, linear processes described by a state-space model of the form

Table 2. Reducing the Controller Tunable Parameters to Two (α and ϵ)

r	β_1	γ_1	β_2	γ_2	β_3	γ_3
1	α	ϵ	—	—	—	—
2	2α	2ϵ	α^2	ϵ^2	—	—
3	3α	3ϵ	$3\alpha^2$	$3\epsilon^2$	α^3	ϵ^3

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t), \quad x(0)=0,\ u(0)=0 \\ y(t) = cx(t-\theta) \end{cases} \tag{16}$$

where A, b and c are $n\times n$, $n\times 1$ and $1\times n$ constant matrices, respectively. This class of systems is a special case of Eq. 1 for $f(x(t)) = Ax(t)$, $g(x(t)) = b$, $h(x(t-\theta)) = cx(t-\theta)$. It is assumed that the delay-free part of the system of Eq. 16 is minimum-phase and has a finite relative order r (smallest integer for which $cA^{r-1}b \neq 0$).

Application of the control law of Eq. 15 to the linear processes of the form of Eq. 16 leads to the linear controller

$$\begin{cases} \dot{\xi} = A_c^*\xi + b_c^* e' \\ \dot{x} = Ax + bu \\ u = \text{sat}\left\{ -\dfrac{1}{\beta_r cA^{r-1}b}\left[c + \sum_{l=1}^{r}\beta_l cA^l\right]x + \dfrac{1}{\gamma_r cA^{r-1}b}[e' + c_c^*\xi]\right\} \end{cases} \tag{17}$$

where $e'(t) = e(t) + cx(t-\theta)$.

Parameterization of the linear control law of Eq. 17 according to the modified IMC structure leads to

$$u(s) = \text{sat}\{w_1(s) - w_2(s)\} = \text{sat}\{Q_1(s)e'(s) - Q_2(s)u(s)\}$$

where the components P, Q_1, and Q_2 are given in Table 3. Note that in this linear case, Q_1 does not depend on the state x, and thus the only input to Q_1 is e'. Furthermore, the transfer functions $P(s)$, $Q_1(s)$, and $Q_2(s)$ are exactly identical to those in the modified IMC parameterization (Zheng et al., 1994), implying that the linear control law of Eq. 17 is a minimal-order state-space realization of a modified IMC controller. When the controller tunable parameters are chosen according to $\beta_l = \gamma_l$, $l = 1, \ldots, r$, the linear control law of Eq. 17 will be exactly a model state-feedback controller.

Example 1. Consider the same first-order linear example and the same unconstrained desired closed-loop response used in (Zheng et al., 1994): $y(s) = [2/(100s+1)]u(s)$, and

$$\frac{y(s)}{y_{sp}(s)} = \frac{1}{20s+1} \tag{18}$$

A minimal-order state-space realization of the process is: $\dot{x} = -0.01x + 0.02u$, $y = x$. For this example, $r = 1$, $cA^{r-1}b = 0.02$, and $\gamma_r = 20$. The transfer functions $Q_1(s) = 2.5$ and $Q_2(s) = 4/(100s+1)$, for $\beta_1 = 20$; $Q_1(s) = 50(s+1)/(20s+1)$ and $Q_2(s) = 99/(100s+1)$, for $\beta_1 = 1$. They are the same as those reported by Zheng et al. (1994). Note that in the absence of constraints, the linear controller induces the same closed-loop response of Eq. 18, irrespective of the value of β_1.

Table 3. Modified IMC Parameterization

$P(s)$	$c(sI-A)^{-1}be^{-\theta s}$
$Q_1(s)$	$\dfrac{1}{\beta_r cA^{r-1}b}\dfrac{\beta_r s^r + \cdots + \beta_1 s + 1}{\gamma_r s^r + \cdots + \gamma_1 s + 1}$
$Q_2(s)$	$\dfrac{1}{\beta_r cA^{r-1}b}[c + \beta_1 cA + \cdots + \beta_r cA^r](sI-A)^{-1}b$

Application to Nonlinear Reactors

Chemical reactor

We consider the same CSTR example and control objective described in the motivation subsection. Application of the control law of Theorem 1 to this reactor leads to the following generalized, mixed error- and state-feedback controller

$$\begin{cases} \dfrac{d\eta_1}{dt} = -\dfrac{1}{\beta_1}\eta_1 + \dfrac{1}{\beta_1}\left[T + \beta_1\left[\dfrac{R_H(\overline{C}_A,\overline{T})}{\rho c} + \dfrac{T_i - \overline{T}}{\tau} + \dfrac{u}{\rho cV}\right]\right], & \eta_1(0) = \overline{T}(0) \\ \dfrac{d\xi_1}{dt} = -\dfrac{1}{\gamma_1}\xi_1 + \dfrac{1}{\gamma_1}(\eta_1 + e), & \xi_1(0) = \overline{T}(0) \\ u = \text{sat}\left\{\dfrac{\rho cV}{\gamma_1}\left(\eta_1 + e + \left[\dfrac{\gamma_1}{\beta_1} - 1\right]\xi_1 - \dfrac{\gamma_1}{\beta_1}\overline{T} - \gamma_1\left[\dfrac{R_H(\overline{C}_A,\overline{T})}{\rho c} + \dfrac{T_i - \overline{T}}{\tau}\right]\right)\right\} \end{cases} \tag{19}$$

Simulation Results. The closed-loop system under the controllers of Eqs. 4 and 19 was simulated in order to examine the respective servo and regulatory performance of the controllers in the presence of the input constraints and to show the considerable improvement in the closed-loop response when the controller of Eq. 19 is used. The following numerical simulations were carried out: reactor startup when $|Q| \leq 10$ kJ·s^{-1}; rejection of unmeasurable disturbances when $|Q| \leq 10$ kJ·s^{-1}; and reactor startup when $-2.5 \leq Q \leq 10$ kJ·s^{-1}.

Startup Performance When $|Q| \leq 10$ kJ·s^{-1}. Figure 5a depicts the startup profiles of the controlled output and manipulated input under the nonlinear controllers of Eqs. 4 and 19 with $\gamma_1 = 100$ s. In the absence of input constraints, the closed-loop responses under both controllers (solid line) were exactly the requested first-order response. In the presence of the input constraints, the controller of Eq. 4 showed a very poor performance (indicating integral windup), while the controller of Eq. 19 (Cases B3, B4 and B5) exhibited significantly better performance (the closed-loop responses in Cases B3, B4 and B5 are free from overshoot and have faster settling times). In the presence of the input constraints and under the controller of Eq. 19, the mismatch between the un-

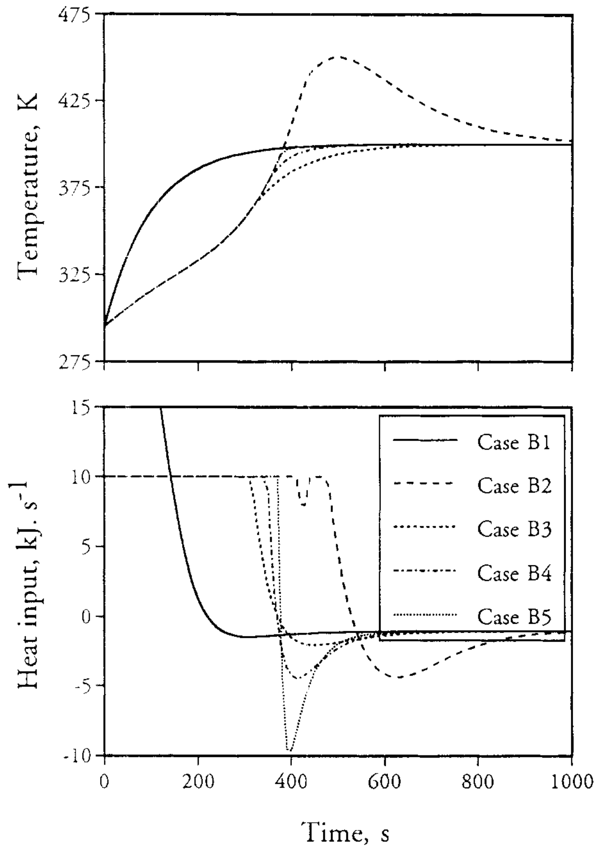

Figure 5a. Startup profiles of the controlled and manipulated variables of the chemical reactor example.

B1, under the controllers of Eqs. 4 and 19 when $|Q| < \infty$; B2, under the controller of Eq. 4 when $|Q| \le 10\ kJ\cdot s^{-1}$; B3, B4, B5, under the controller of Eq. 19 with $\beta_1 = 100$, 40, 10 s, respectively, and when $|Q| \le 10\ kJ\cdot s^{-1}$.

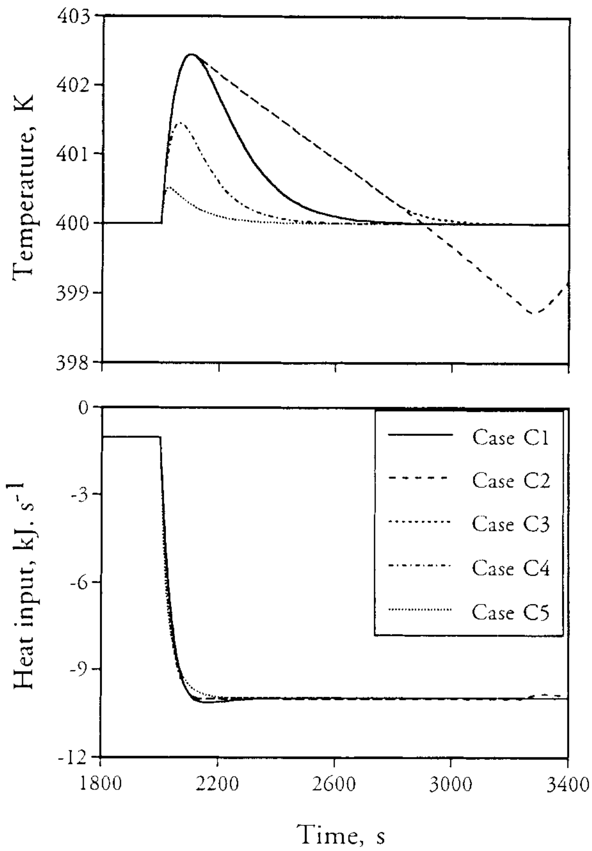

Figure 5b. Profiles of the controlled and manipulated variables of the chemical reactor example in the presence of the unmeasurable disturbances.

C1, under the controllers of Eqs. 4 and 19 when $|Q| < \infty$; C2, under the controller of Eq. 4 when $|Q| \le 10\ kJ\cdot s^{-1}$; C3, C4, C5, under the controller of Eq. 19 with $\beta_1 = 100$, 40, 10 s, respectively, and when $|Q| \le 10\ kJ\cdot s^{-1}$.

constrained and constrained closed-loop responses was minimized by decreasing the value of β_1. The heat input rate stayed at its upper limit the most when $\beta_1 = 10$ s. These simulation results clearly demonstrated the improvement that can be made in the closed-loop response, by using the generalized, mixed error- and state-feedback controller of Eq. 19.

Regulatory Performance When $|Q| \le 10\ kJ\cdot s^{-1}$. In order to show the regulatory performance of the generalized, mixed error- and state-feedback controller of Eq. 19, simultaneous step changes of +20 K and $+4\ kmol\cdot m^{-3}$ were introduced in the inlet temperature T_i and in the inlet concentration C_{A_i}, respectively, when the reactor was operating at the steady state. These disturbances were considered to be unmeasurable. Figure 5b depicts the corresponding profiles of the controlled output and manipulated input. The relative regulatory performance of the two controllers was very much similar to their relative startup performance: the generalized, mixed error- and state-feedback controller with $\beta_1 = 10$ s exhibited the "best" regulatory performance.

Startup Performance When $-2.5 \le Q \le 10\ kJ\cdot s^{-1}$. Figure 5c depicts the startup profiles of the controlled output and manipulated input under the nonlinear controllers of Eqs. 4 and 19 with $\gamma_1 = 50$ s. In the absence of input constraints, the closed-loop performance under both controllers (solid line) was exactly the requested first-order response. In the presence of the input constraints and under both controllers, the upper and lower constraints were respectively active during the transient period. The controller of Eq. 4 again exhibited a very poor performance (indicating integral windup), and the controller of Eq. 19 showed much better responses (with less overshoot and faster settling times), for all the three values of β_1.

However, unlike in the case that $|Q| \le 10\ kJ\cdot s^{-1}$, the performance of the controller of Eq. 19 improved as the value of β_1 increased in this case. The closed-loop responses corresponding to the smaller values of β_1 ($\beta_1 = 10$, 50 s) had overshoots. Both these overshoots were due to shortsightedness of the analytical controller (Walgama et al., 1992; Coulibaly et al., 1995): for small values of β_1, the controller of Eq. 19 forced the constrained response to reach the linear unconstrained response in a very short period of time by requesting excessive heat (longer stay of the heat input rate at its upper limit), while it was totally unaware that the lower constraint would be active in the near future. Because of this overheating together with the higher rate of heat production by the reactions at the higher temperatures, the controller then demanded excessive infeasible cooling, leading to a long stay of the heat input rate at the lower constraint and to the over-

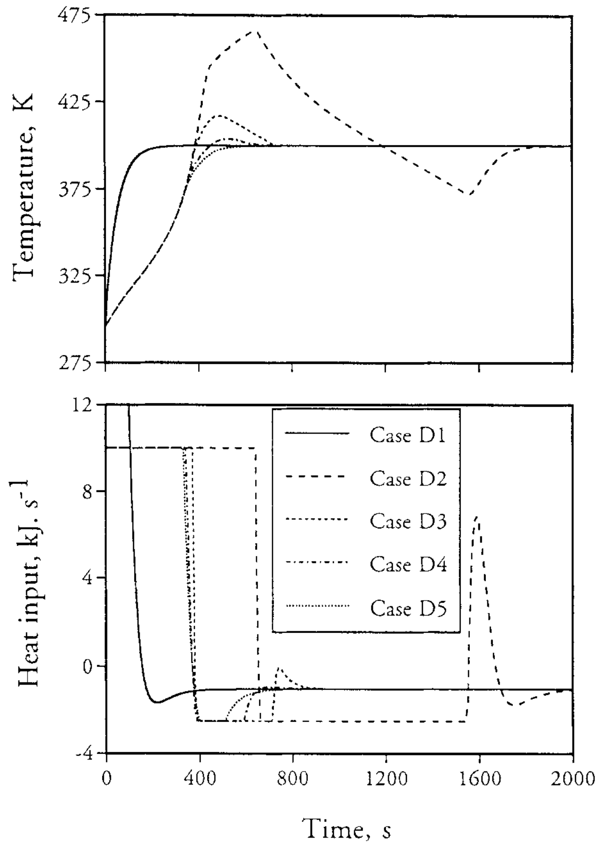

Figure 5c. Startup profiles of the controlled and manipulated variables of the chemical reactor example.

D1, under the controllers of Eqs. 4 and 19 when $|Q| < \infty$; D2, under the controller of Eq. 4 when $-2.5 \leq Q \leq 10$ kJ·s^{-1}; D3, D4, D5, under the controller of Eq. 19 with $\beta_1 =$ 10, 50, 70 s, respectively, and when $-2.5 \leq Q \leq 10$ kJ·s^{-1}.

shoots. The shortsightedness problem was aptly overcome by setting $\beta_1 = 70$ s, leading to a closed-loop response with lower settling time and no overshoot (dotted line).

Biochemical reactor

Consider a continuous stirred-tank bioreactor represented by the following:

$$\begin{cases} \dfrac{d\bar{S}}{dt} = -\dfrac{1}{Y_{X/S}}\mu(\bar{S})\bar{X} + D\left(S_f - \bar{S}\right) \\ \dfrac{d\bar{X}}{dt} = \mu(\bar{S})\bar{X} - D\bar{X} \end{cases} \tag{20}$$

where the specific growth rate $\mu(\bar{S})$ (s^{-1}) is given by

$$\mu(\bar{S}) = \frac{\mu_m \bar{S}}{K_S + \bar{S} + \dfrac{\bar{S}^2}{K_I}}$$

The operating conditions and model parameters of the reactor are given in Table 4. The control objective is to perform reactor startup and operate the reactor at the asymptotically stable steady state $(S_{ss}, X_{ss}) = (2.22, 3.88)$ (kg·m^{-3}) by controlling the substrate concentration in the reactor $\bar{S}$ and by manipulating the dilution rate D (s^{-1}). This steady state corresponds to the maximum cell productivity $(D\bar{X})$ of 7.03×10^{-4} kg·m^{-3}·s^{-1}. It is assumed that: the dilution rate is bounded ($0 \leq D \leq 1.94 \times 10^{-4}$s^{-1}); the substrate concentration is measured with a time delay of 1,080 s ($\theta = 1{,}080$ s); and the cell concentration is not measured.

Table 4. Operating Conditions and Model Parameters of the Bioreactor Example

$S_f = 1.00 \times 10^1$	kg·m^{-3}
$S(0) = 0.00 \times 10^0$	kg·m^{-3}
$X(0) = 1.00 \times 10^{-2}$	kg·m^{-3}
$S_{ss} = 2.22 \times 10^0$	kg·m^{-3}
$X_{ss} = 3.89 \times 10^0$	kg·m^{-3}
$D_{ss} = 1.81 \times 10^{-4}$	s^{-1}
$K_S = 1.00 \times 10^0$	kg·m^{-3}
$K_I = 2.50 \times 10^1$	kg·m^{-3}
$\mu_m = 2.78 \times 10^{-4}$	s^{-1}
$Y_{X/S} = 5.00 \times 10^{-1}$	

The process model can be recast in the form of Eq. 1 with $y = S(t-1{,}080)$, $x = [S\ X]^T$, $u = D$. For this problem, $r = 1$ $(L_g h = (S_f - S) \neq 0)$.

- *Generalized, Reduced-Order, Error-Feedback Controller.* Application of the control law of Theorem 2 to this bioreactor results in

$$\begin{cases} \dfrac{dS}{dt} = -\dfrac{1}{Y_{X/S}}\mu(S)X + u(S_f - S), & S(0) = 0 \\ \dfrac{dX}{dt} = \mu(S)X - uX, & X(0) = 0.01 \\ \dfrac{d\xi_1}{dt} = -\dfrac{1}{\gamma_1}\xi_1 + \dfrac{1}{\gamma_1}e', & \xi_1(0) = 0 \\ u = \text{sat}\left\{ \dfrac{e' + \left(\dfrac{\gamma_1}{\beta_1} - 1\right)\xi_1 - \dfrac{\gamma_1}{\beta_1}S + \gamma_1 \dfrac{1}{Y_{X/S}}\mu(S)X}{\gamma_1(S_f - S)} \right\} \end{cases} \tag{21}$$

where $u_{\min} = 0$ s^{-1}, $u_{\max} = 1.94 \times 10^{-4}$ s^{-1}, and $e'(t) = e(t) + S(t-1{,}080)$.

- *"Nonlinear IMC" Controller.* The following controller is a nonlinear IMC controller parameterized according to the original IMC structure

$$P: \begin{cases} \dfrac{dS}{dt} = -\dfrac{1}{Y_{X/S}}\mu(S)X + (S_f - S)\,\text{sat}\{w\}, & S(0) = 0 \\ \dfrac{dX}{dt} = \mu(S)X - X\,\text{sat}\{w\}, & X(0) = 0.01 \end{cases}$$

$$Q: \begin{cases} \dfrac{d\lambda_1}{dt} = -\dfrac{1}{Y_{X/S}}\mu(\lambda_1)\lambda_2 + (S_f - \lambda_1)w, & \lambda_1(0) = 0 \\ \dfrac{d\lambda_2}{dt} = \mu(\lambda_1)\lambda_2 - \lambda_2 w, & \lambda_2(0) = 0.01 \\ w = \dfrac{e' - \lambda_1 + \gamma_1 \dfrac{1}{Y_{X/S}}\mu(\lambda_1)\lambda_2}{\gamma_1(S_f - \lambda_1)} \end{cases} \tag{22}$$

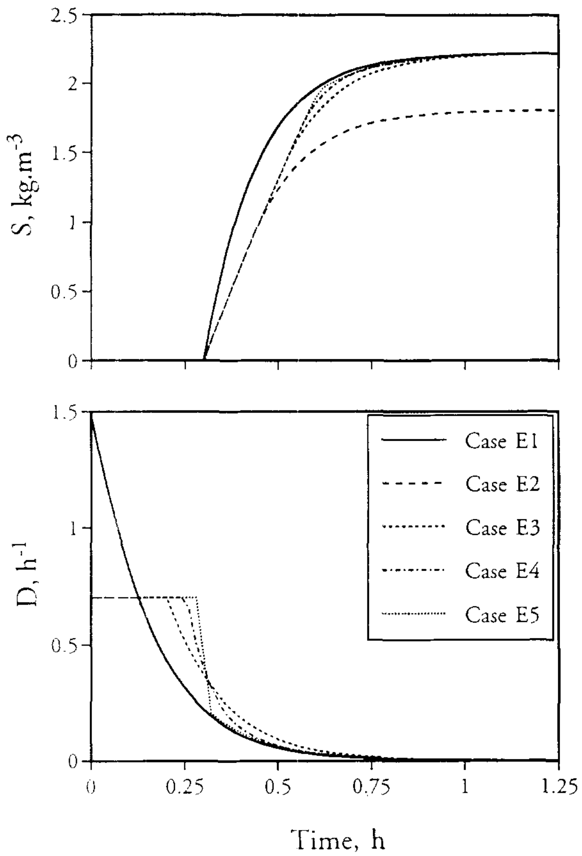

Figure 6a. Startup profiles of the controlled and manipulated variables of the bioreactor example.

E1, under the controllers of Eqs. 21 and 22 when $|D| < \infty$; E2, under the controller of Eq. 22 when $0 \le D \le 1.94 \times 10^{-4}$ s^{-1}; E3, E4, E5, under the controller of Eq. 21 with $\beta_1 = 540, 216, 72$ s, respectively, when $0 \le D \le 1.94 \times 10^{-4}$ s^{-1}.

where $u_{min} = 0$ s^{-1}, $u_{max} = 1.94 \times 10^{-4}$ s^{-1}, $e'(t) = e(t) + S(t - 1{,}080)$, P represents the process model wherein the state variables S and X are driven by sat(w), and Q represents the process model inverse plus the IMC filter $[1/(\gamma_1 s + 1)]$, wherein the state variables λ_1 and λ_2 are driven by w.

Simulation Results. The closed-loop system under the controllers of Eqs. 21 and 22 with $\gamma_1 = 540$ s was simulated in order to examine the servo and regulatory performance of the controllers in the presence of the input constraints and deadtime. The numerical simulations include reactor startup and constant, unmeasurable, disturbance rejection, by manipulating the bounded dilution rate, $0 \le D \le 1.94 \times 10^{-4}$ s^{-1}.

Startup Performance. Figure 6a depicts the startup profiles of the controlled output and manipulated input. In the absence of input constraints, the closed-loop response under both controllers (solid line) exactly followed the requested first-order response. In the presence of the input constraints and under the controller of Eq. 21, the difference between the unconstrained and constrained closed-loop responses was minimized by decreasing the value of β_1; this trend is very much similar to that in the CSTR example with a single active input constraint. Indeed, for all the selected values of β_1', the respective closed-loop responses were close to that of the unconstrained case (solid line) and were much better than the closed-loop response obtained using the nonlinear IMC controller (long-dashed line). The closed-loop response under the nonlinear IMC controller was extremely sluggish (had an extremely long response time). This very poor performance was due to that as pointed out by Zheng et al. (1994), the original IMC simply "clips" the unconstrained controller action (see the manipulated variable profile indicated by the long-dashed line). It was also a consequence of the fact that the process model inverse plus filter (Q component of the controller of Eq. 22) was unaware of the active input constraint and that there were significant discrepancies between S and λ_1 and between X and λ_2.

Regulatory Performance. In order to ascertain the regulatory performance of the generalized, reduced-order, error-feedback controller of Eq. 21, when the reactor was operating at steady state, a step change of -10% was introduced in the substrate feed concentration S_f. This disturbance was considered to be unmeasurable. Figure 6b depicts the profiles of the controlled output and manipulated input under the generalized, reduced-order, error-feedback controller of Eq. 21 and the nonlinear IMC controller of Eq. 22. The relative regulatory performance of the two controllers was very much similar to their relative startup performance, and the generalized, error-feedback controller with $\beta_1 = 7.2 \times 10^1$ s exhibited the "best" regulatory performance.

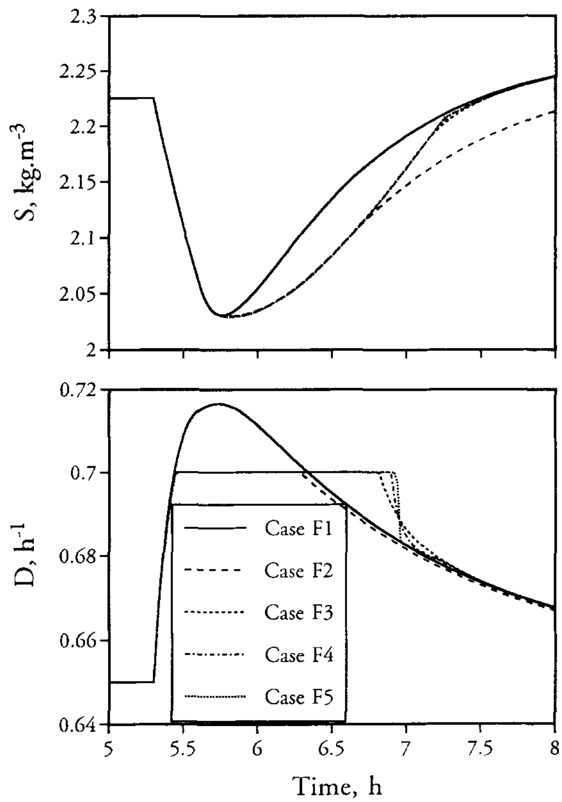

Figure 6b. Profiles of the controlled and manipulated variables of the bioreactor example in the presence of the unmeasurable disturbance.

F1, under the controllers of Eqs. 21 and 22 when $|D| < \infty$; F2, under the controller of Eq. 22 when $0 \le D \le 1.94 \times 10^{-4}$ s^{-1}; F3, F4, F5, under the controller of Eq. 21 with $\beta_1 = 540, 216, 72$ s, respectively, when $0 \le D \le 1.94 \times 10^{-4}$ s^{-1}.

Conclusions

An optimization-based method for analytical control of nonlinear processes with input constraints was presented. It led to the development of two analytical control laws that (a) can minimize the mismatch between the constrained closed-loop response and the nominal, linear, unconstrained, closed-loop response and (b) inherently compensate for windup. The adjustable parameters of the control laws offer great flexibility to obtain a desirable closed-loop response in the presence of active input constraints. The control laws allow one to adjust:

- The time period during which an input constraint is active;
- The shape of the closed-loop response, without changing the shape of the nominal linear unconstrained response, during the time period within which (a) none of the constraints are active and (b) the response has not reached the nominal linear unconstrained response.

Furthermore, in the absence of input constraints, these control laws are input-output linearizing. The improvement in closed-loop performance gained by using the derived control laws was demonstrated by numerical simulations of chemical and biochemical reactor examples. The connections between the developed control laws and the modified internal model control were established.

While tunable parameters $\gamma_1, \ldots, \gamma_r$ determine the shape of the linear unconstrained closed-loop response, y^* (described by Eq. 5), the tunable parameters $\beta_1, \ldots, \beta_r$

- determine the rate of decay (and the shape of the profile) of the mismatch between the constrained output response y and the linear unconstrained closed-loop response, when none of the constraints are active.
- do not affect the shape of the linear unconstrained closed-loop response.

As demonstrated by the chemical reactor example, if during a transient period only an upper or lower limit is active (as in Figure 6a), then a better closed-loop response (with a lower ISE) is obtained by placing all the roots of the polynomial of Eq. 14 further left in the complex plane. However, if during a transient period both limits are active (see the results shown in Figure 6c), then a better closed-loop response (with a lower ISE) may be obtained by placing all the roots of the polynomial Eq. 14 closer to the origin. The fact is that the derived control laws minimize the performance index of Eq. 6 (over a sufficiently short horizon) in the limit that all the roots of the polynomial of Eq. 14 are placed far left in the complex plane, and that minimization of the performance index of Eq. 6 over a very short horizon may not lead to the minimization of the performance index of Eq. 6 over an infinite horizon. Of course, a minimization over a long horizon is more favorable but does not have an analytical solution in general.

This article presented the basic idea behind the optimization-based method for handling input constraints in input-output linearizing control. This method has already been extended to multi-input multi-output (MIMO) processes (Valluri, 1997). The derived MIMO control laws consist of two distinct components: (i) an input-output linearizing controller that inherently compensates for windup and (ii) an optimal directionality compensator. In the case that the characteristic (decoupling) matrix of process is diagonal, the optimal directionality compensator is identical to "clipping." For general processes, however, neither clipping nor direction preservation can optimally compensate for process directionality (Valluri, 1997). The MIMO results will be presented in a forthcoming article. The controlled laws presented in this article are applicable to

- Single-input single-output processes with input and output delays: a process with input and output delays can be recast in the form of Eq. 1 (Soroush and Kravaris, 1996)
- Processes with nonminimum-phase delay-free part, via using a "minimum-phase" statically-equivalent output (see Wright and Kravaris, 1992), for a definition of a minimum-phase statically-equivalent output).

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Notation

A = reactant
c = heat capacity of reacting mixture, $kJ \cdot kg^{-1} \cdot K^{-1}$
$C_A, \overline{C}_A$ = model-predicted and measured values of the reactant concentration, $kmol \cdot m^{-3}$
C_{A_i} = inlet concentration of reactant A, $kmol \cdot m^{-3}$
D_{ss} = steady-state dilution rate, s^{-1}
K_I = substrate inhibition constant, $kg \cdot m^{-3}$
K_S = saturation constant, $kg \cdot m^{-3}$
R_A = rate of production of A, $kmol \cdot m^{-3} \cdot s^{-1}$
R_H = overall rate of heat production by reactions, $kJ \cdot m^{-3} \cdot s^{-1}$
$S, \overline{S}$ = model-predicted and measured values of the substrate concentration, $kg \cdot m^{-3}$
S_f = substrate concentration in the feed, $kg \cdot m^{-3}$
T_i = temperature of inlet stream, K
V = volume of the reacting mixture, m^3
$X, \overline{X}$ = model-predicted and measured values of the cell concentration, $kg \cdot m^{-3}$
X_{ss} = steady-state cell concentration, $kg \cdot m^{-3}$
$\hat{y}$ = predicted value of the controlled variable
$Y_{X/S}$ = yield coefficient
β_j = controller tunable parameter ($j = 1, \ldots, r$), s^j
γ_j = controller tunable parameter ($j = 1, \ldots, r$), s^j
μ_m = maximum specific growth rate, s^{-1}
ρ = density of reacting mixture, $kg \cdot m^{-3}$
τ = CSTR residence time, s; time, s

Math symbols

int = interior
$L_f h(x)$ = Lie derivative of the scalar field $h(x)$ with respect to the vector field $f(x)$
$= L_f^1 h(x) \overset{\Delta}{=} \sum_{l=1}^{n} [\partial h(x)/\partial x_l] f_l(x)$
$L_f^{l+1} h(x)$ = Lie derivative of scalar field $L_f^l h(x)$ with respect to vector field $f(x)$
$L_g L_f^l h(x)$ = Lie derivative of scalar field $L_f^l h(x)$ with respect to vector field $g(x)$
$\|\omega(t)\|_p$ = the p-function norm of $\omega(t)$ over a finite time interval $[a, b]$, where $p \geq 1$
$= (\int_a^b |\omega(t)|^p dt)^{1/p}$

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Appendix A: Representative Matrix Triplets (A_c, b_c, c_c) and (A_c^*, b_c^*, c_c^*)

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{1}{\beta_r} & -\dfrac{\beta_1}{\beta_r} & -\dfrac{\beta_2}{\beta_r} & \cdots & -\dfrac{\beta_{r-1}}{\beta_r} \end{bmatrix},$$

$$b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{\beta_r} \end{bmatrix}, \qquad c_c = [1 \quad 0 \quad \cdots \quad 0]$$

$$A_c^* = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{1}{\gamma_r} & -\dfrac{\gamma_1}{\gamma_r} & -\dfrac{\gamma_2}{\gamma_r} & \cdots & -\dfrac{\gamma_{r-1}}{\gamma_r} \end{bmatrix}, \quad b_c^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{\gamma_r} \end{bmatrix}$$

$$c_c^* = \left[\left(\gamma_r \frac{1}{\beta_r} - 1 \right) \quad \left(\gamma_r \frac{\beta_1}{\beta_r} - \gamma_1 \right) \quad \cdots \quad \left(\gamma_r \frac{\beta_{r-1}}{\beta_r} - \gamma_{r-1} \right) \right]$$

Appendix B: Proofs

Proof of Theorem 1

Part (a): Upon substitution for matrices A_c, b_c, and c_c (see the Appendix A), the control law of Theorem 1 takes the form

$$\begin{cases} \dot{\xi} = A_c^* \xi + b_c^* (e + \eta_1) \\ \dot{\eta}_1 = \eta_2 \\ \vdots \\ \dot{\eta}_{r-1} = \eta_r \\ \dot{\eta}_r = -\dfrac{1}{\beta_r}\eta_1 - \dfrac{\beta_1}{\beta_r}\eta_2 - \cdots - \dfrac{\beta_{r-1}}{\beta_r}\eta_r + \dfrac{1}{\beta_r}\Phi(\bar{x}, u) \\ u = \text{sat}\{\Psi_0(\bar{x}, \eta_1 + e + c_c^* \xi)\} \end{cases} \tag{B1}$$

where

$$\Phi(\bar{x}, u) = h(\bar{x}) + \sum_{l=1}^{r} \beta_l L_f^l h(\bar{x}) + \beta_r L_g L_f^{r-1} h(\bar{x}) u$$

$$= y + \beta_1 \frac{dy}{dt} + \cdots + \beta_r \frac{d^r y}{dt^r}$$

Thus,

$$\beta_r \dot{\eta}_r = -\eta_1 - \beta_1 \eta_2 - \cdots - \beta_{r-1}\eta_r + \left[y + \beta_1 \frac{dy}{dt} + \cdots + \beta_r \frac{d^r y}{dt^r} \right]$$

$$(y - \eta_1) + \beta_1 \frac{d(y - \eta_1)}{dt} + \cdots + \beta_r \frac{d^r(y - \eta_1)}{dt^r} = 0$$

Since it was assumed that $\eta_{l+1}(0) = L_f^l h[\bar{x}(0)] = d^l y(0)/dt^l$, $l = 0, \ldots, r-1$, the preceding ordinary differential equation has a trivial solution: $\eta_1(t) = y(t)$, $\forall t \geq 0$. Thus,

$$c_c\eta(t)+e(t)=\eta_1(t)+y_{sp}(t)-\bar{y}(t)=y_{sp}(t)-d(t),\ \forall t\geq 0$$

Using the relationship $c_c\eta(t)+e(t)=y_{sp}(t)-d(t)$, the controller of Eq. B1 is simplified to

$$\begin{cases}\dot{\xi}=A_c^*\xi+b_c^*[y_{sp}-d]\\ u=\text{sat}\left\{\dfrac{\dfrac{\beta_r}{\gamma_r}[y_{sp}-d+c_c^*\xi]-h(\bar{x})-\sum_{l=1}^{r}\beta_l L_f^l h(\bar{x})}{\beta_r L_g L_f^{r-1}h(\bar{x})}\right\}\end{cases}\tag{B2}$$

By the identity $d=\bar{y}-h(\bar{x})$, we see that the preceding dynamic system consists of the static state feedback

$$u=\text{sat}\left\{\frac{\zeta-[\bar{y}-h(\bar{x})]-h(\bar{x})-\sum_{l=1}^{r}\beta_l L_f^l h(\bar{x})}{\beta_r L_g L_f^{r-1}h(\bar{x})}\right\}\tag{B3}$$

and the dynamic subsystem

$$\begin{cases}\dot{\xi}=A_c^*\xi+b_c^*[y_{sp}-d]\\ \zeta=\dfrac{\beta_r}{\gamma_r}c_c^*\xi+\dfrac{\beta_r}{\gamma_r}[y_{sp}-d]+d\end{cases}\tag{B4}$$

which is a minimal-order state-space realization of

$$[\zeta-d]+\gamma_1\frac{d[\zeta-d]}{dt}+\cdots+\gamma_r\frac{d^r[\zeta-d]}{dt^r}=[y_{sp}-d]+\beta_1\frac{d[y_{sp}-d]}{dt}+\cdots+\beta_r\frac{d^r[y_{sp}-d]}{dt^r}$$

In the limit that all the roots of the polynomial $\beta_r s^r+\cdots+\beta_1 s+1=0$ are placed far left in the complex plane, (that is, in the limit that $\beta_l\to 0$, $l=1,\ \ldots,\ r$), (i) ζ will be governed by

$$\zeta+\gamma_1\frac{d\zeta}{dt}+\cdots+\gamma_r\frac{d^r\zeta}{dt^r}=y_{sp}\tag{B5}$$

that is, $\zeta=y^*$, where y^* is the linear unconstrained closed-loop output response governed by Eq. 5, and (ii) according to Theorem 1 given in (Valluri et al., 1997), the static state feedback of Eq. B3 will be the solution to the one-dimensional quadratic optimization problem

$$\min_{u(t)}\left\{\|\zeta(\tau)-\hat{y}(\tau)\|_P^2\right\}=\min_{u(t)}\left\{\|y^*(\tau)-\hat{y}(\tau)\|_P^2\right\}\tag{B6}$$

subject to the constraints $u_{\min}\leq u(t)\leq u_{\max}$ for a sufficiently short horizon T_h.

Part (b): Using the relationship $c_c\eta+e=y_{sp}-d$ proved in Part (a), we see that in the absence of the constraints, the control law of Eq. B1 takes the form

$$\begin{cases}\dot{\xi}=A_c^*\xi+b_c^*[y_{sp}-d]\\ u=\dfrac{y_{sp}-d+\left(\dfrac{\gamma_r}{\beta_r}-1\right)\xi_1+\sum_{l=1}^{r-1}\left(\gamma_r\dfrac{\beta_l}{\beta_r}-\gamma_l\right)\xi_{l+1}-\dfrac{\gamma_r}{\beta_r}h(\bar{x})-\sum_{l=1}^{r}\gamma_r\dfrac{\beta_l}{\beta_r}L_f^l h(\bar{x})}{\gamma_r L_g L_f^{r-1}h(\bar{x})}\end{cases}\tag{B7}$$

The ξ subsystem (see the Appendix A) is a minimal-order state-space realization of Eq. 10, and thus in the absence of the constraints

$$\xi_{l+1}=\frac{d^l y}{dt^l}=L_f^l h(\bar{x}),\qquad l=0,\ \ldots,\ r-1\tag{B8}$$

Substituting for ξ_l, $l=1,\ \ldots,\ r$, in the state feedback of Eq. B7, we obtain

$$u=\frac{y_{sp}-d-h(\bar{x})-\sum_{l=1}^{r}\gamma_l L_f^l h(\bar{x})}{\gamma_r L_g L_f^{r-1}h(\bar{x})}$$

which induces the linear input-output response of Eq. 5 with $\theta=0$.

Part (c): When the closed-loop control system is asymptotically stable (see Theorem 3) and process is subjected to "rejectable" (in the sense that always $u_{ss}\in\text{int}(U)$) constant disturbances and model errors, the closed-loop system reaches a steady state (an equilibrium point), and therefore

$$\eta_{1_{ss}}=\Phi(x_{ss},u_{ss}),\quad \eta_{l_{ss}}=0,\ l=2,\ \ldots,\ r,$$

$$\xi_{1_{ss}}=\eta_{1_{ss}}+e_{ss},\quad \xi_{l_{ss}}=0,\ l=2,\ \ldots,\ r,$$

and

$$u_{ss}=\Psi_0\left[x_{ss},\ \Phi(x_{ss},\ u_{ss})+e_{ss}+\left(\frac{\gamma_r}{\beta_r}-1\right)\xi_{1_{ss}}\right]$$

The preceding equation can be simplified, leading to

$$u_{ss}=\Psi_0\left[x_{ss},\ \frac{\gamma_r}{\beta_r}\{\Phi(x_{ss},\ u_{ss})+e_{ss}\}\right]$$

and

$$u_{ss}=\frac{\frac{\gamma_r}{\beta_r}\left[h(x_{ss})+\sum_{l=1}^{r}\beta_l L_f^l h(x_{ss})+\beta_r L_g L_f^{r-1}h(x_{ss})u_{ss}+e_{ss}\right]-\frac{\gamma_r}{\beta_r}\left[h(x_{ss})+\sum_{l=1}^{r}\beta_l L_f^l h(x_{ss})\right]}{\gamma_r L_g L_f^{r-1}h(x_{ss})},$$

which implies that $e_{ss}=0$.

Proof of Theorem 2

Part (a): In the absence of model errors and disturbances, $e'=y_{sp}-d$, and in this case, the control law of Theorem 2 is simplified to

$$\begin{cases}\dot{\xi}=A_c^*\xi+b_c^*[y_{sp}-d]\\ \dot{x}=f(x)+g(x)u\\ u=\text{sat}\left\{\dfrac{\frac{\beta_r}{\gamma_r}[y_{sp}-d+c_c^*\xi]-h(x)-\sum_{l=1}^{r}\beta_l L_f^l h(x)}{\beta_r L_g L_f^{r-1}h(x)}\right\}\end{cases}\tag{B9}$$

which consists of the dynamic feedback

$$\begin{cases}\dot{x}=f(x)+g(x)u\\ u=\text{sat}\left\{\dfrac{\zeta-d-h(x)-\sum_{l=1}^{r}\beta_l L_f^l h(x)}{\beta_r L_g L_f^{r-1}h(x)}\right\}\end{cases}\tag{B10}$$

and the dynamic subsystem of Eq. B4. In the limit that all the roots of the polynomial $\beta_r s^r+\cdots+\beta_1 s+1=0$ are placed far left in the complex plane, (i) ζ will be governed again by Eq. B5, i.e., $\zeta=y^*$; and (ii) according to Theorem 2 given in (Valluri et al., 1997), the dynamic system of Eq. B10 will be the solution to the one-dimensional quadratic optimization problem of Eq. B6 for a sufficiently short horizon T_h.

Part (b): Using the relationship $e'=y_{sp}-d$, we see that in the absence of constraints, the control law of Eq. B9 takes the form

$$\begin{cases}\dot{\xi}=A_c^*\xi+b_c^*[y_{sp}-d]\\ \dot{x}=f(x)+g(x)u\\ u=\dfrac{y_{sp}-d+\left(\frac{\gamma_r}{\beta_r}-1\right)\xi_1+\sum_{l=1}^{r-1}\left(\gamma_r\frac{\beta_l}{\beta_r}-\gamma_l\right)\xi_{l+1}-\frac{\gamma_r}{\beta_r}h(x)-\sum_{l=1}^{r}\gamma_r\frac{\beta_l}{\beta_r}L_f^l h(x)}{\gamma_r L_g L_f^{r-1}h(x)}\end{cases}\tag{B11}$$

and from Eq. B8, substituting for ξ_l, $l=1,\ldots,r$, in the dynamic system of Eq. B11, we obtain

$$\begin{cases}\dot{x}=f(x)+g(x)u\\ u=\dfrac{y_{sp}-d-h(x)-\sum_{l=1}^{r}\gamma_l L_f^l h(x)}{\gamma_r L_g L_f^{r-1}h(x)}\end{cases}$$

which induces the linear input-output response of Eq. 5.

Part (c): When the closed-loop control system is asymptotically stable (see Theorem 4) and process is subjected to "rejectable" (in the sense that always $u_{ss}\in\text{int}(U)$) constant disturbances and model errors, the closed-loop system reaches a steady state (an equilibrium point), and therefore

$$\xi_{1_{ss}}=e'_{ss},\quad \xi_{l_{ss}}=0,\quad l=2,\ldots,r,$$

and

$$u_{ss}=\Psi_0\left[x_{ss},e'_{ss}+\left(\frac{\gamma_r}{\beta_r}-1\right)e'_{ss}\right]$$

The preceding equation can be simplified, leading to

$$u_{ss}=\Psi_0\left[x_{ss},\frac{\gamma_r}{\beta_r}(e_{ss}+h(x_{ss}))\right],$$

which by $L_f^l h(x_{ss})=0$, $l=1,\ldots,r-1$, and $L_f^r h(x_{ss})+L_g L_f^{r-1}h(x_{ss})u_{ss}=0$, implies that $e_{ss}=0$.

Proof of Theorem 3

Part (a): For a process of the form of Eq. 1 with complete state measurements ($\bar{x}$) and no deadtimes, in the absence of input constraints the closed-loop system under the control law of Eq. 13 is given by

$$\begin{cases}\dot{\bar{x}}=f(\bar{x})+g(\bar{x})\Psi_0(\bar{x},c_c\eta+e+c_c^*\xi)\\ \dot{\eta}=A_c\eta+b_c\Phi(\bar{x},u)\\ \dot{\xi}=A_c^*\xi+b_c^*[c_c\eta+e]\\ u=\Psi_0(\bar{x},c_c\eta+e+c_c^*\xi)\end{cases}\tag{B12}$$

where

$$\Phi(\bar{x},u)=h(\bar{x})+\sum_{l=1}^{r}\beta_l L_f^l h(\bar{x})+\beta_r L_g L_f^{r-1}h(\bar{x})u=\frac{\beta_r}{\gamma_r}[c_c\eta+e+c_c^*\xi]$$

Let $\Theta = [h(\bar{x})\ L_f h(\bar{x})\ \cdots\ L_f^{r-1}h(\bar{x})]^T$, then the system $\dot{\bar{x}} = f(\bar{x}) + g(\bar{x})\Psi_0(\bar{x}, c_c\eta + e + c_c^*\xi)$ can be recast in the normal form

$$\begin{cases} \dot{\delta} = \phi(\delta, \Theta), & \delta(0) = 0 \\ \dot{\Theta} = A_c\Theta + \dfrac{\beta_r}{\gamma_r} b_c[c_c\eta + e + c_c^*\xi], & \Theta(0) = 0 \end{cases} \tag{B13}$$

where the first subsystem of the system of Eq. B13 is the process forced zero dynamics. Using the change of variable $\Delta = \Theta - \eta$, eliminating Θ from Eq. B13, and setting $c_c\eta + e = \eta_1 + y_{sp} - h(\bar{x}) - d = y_{sp} - d - c_c\Delta$, we recast the closed-loop system of Eq. B12 in the form

$$\begin{cases} \dot{\Delta} = A_c\Delta \\ \dot{\xi} = -b_c^* c_c\Delta + A_c^*\xi + b_c^*[y_{sp} - d] \\ \dot{\eta} = -b_c^* c_c\Delta + b_c^* c_c^*\xi + A_c\eta + b_c^*[y_{sp} - d] \\ \dot{\delta} = \phi(\delta, \Delta + \eta) \end{cases} \tag{B14}$$

The Jacobian matrix of the preceding system is lower block-triangular, and thus the system will be locally asymptotically stable, if all the eigenvalues of the matrices A_c and A_c^* and all the eigenvalues of the Jacobian matrix of the system

$$\dot{\delta} = \phi(\delta, \Delta + \eta) \tag{B15}$$

evaluated at $(\delta, \Delta + \eta) = (0, 0)$, lie in the left half of the complex plane. The eigenvalues of the matrices A_c^* and A_c are the roots of the equations $\gamma_r s^r + \cdots + \gamma_1 s + 1 = 0$ and $\beta_r s^r + \cdots + \beta_1 s + 1 = 0$, respectively, which according to the conditions c3 and c4 all lie in the left half of the complex plane. All the eigenvalues of the Jacobian of the system of Eq. B17 evaluated at $(\delta, \Delta + \eta) = (0, 0)$ also lie in the left half of the complex plane, since the delay-free part of the process was assumed to be minimum-phase (condition c1 holds).

Part (b): By placing a root of the equation $\beta_r s^r + \cdots + \beta_1 s + 1 = 0$ sufficiently close to the origin, according to Eq. 7, the controller eliminates the mismatch between y and y^* so slowly that the plant input may saturate only for an infinitesimal period of time. After this very short saturation period, the mismatch between y and y^* decays according to Eq. 7 and the closed-loop dynamics will be governed by the system of Eq. B12 which in Part (a) of this Theorem was proved to be asymptotically stable when the conditions c1, c3 and c4 hold.

Proof of Theorem 4

Part (a): For a process of the form of Eq. 1 with incomplete state measurements, in the absence of input constraints the closed-loop system under the control law of Eq. 15 is given by

$$\begin{cases} \dot{\bar{x}} = f(\bar{x}) + g(\bar{x})\Psi_0(x, e' + c_c^*\xi) \\ \dot{x} = f(x) + g(x)\Psi_0(x, e' + c_c^*\xi) \\ \dot{\xi} = A_c^*\xi + b_c^* e' \end{cases} \tag{B16}$$

To prove the (local) asymptotic stability of the preceding closed-loop system, we invoke the separation principle from linear control theory. The control law of Eq. 15 includes an open-loop observer, $\dot{x} = f(x) + g(x)u$, whose error dynamics are asymptotically stable (because the process was assumed to be asymptotically stable (condition c2)). Thus, if we prove that

$$\begin{cases} \dot{\bar{x}} = f(\bar{x}) + g(\bar{x})\Psi_0(\bar{x}, e' + c_c^*\xi) \\ \dot{\xi} = A_c^*\xi + b_c^* e' \end{cases} \tag{B17}$$

where $e' = y_{sp} - d$, is asymptotically stable, then the proof of the local asymptotic stability of the closed-loop system of Eq. B16 will be complete.

As in the proof of Theorem 3, we recast the system $\dot{\bar{x}} = f(\bar{x}) + g(\bar{x})\Psi_0(\bar{x}, y_{sp} - d + c_c^*\xi)$ in the normal form

$$\begin{cases} \dot{\delta} = \phi(\delta, \Theta) \\ \dot{\Theta} = A_c\Theta + b_c^*[y_{sp} - d + c_c^*\xi] \end{cases} \tag{B18}$$

where $\Theta = [h(\bar{x})\ L_f h(\bar{x})\ \cdots\ L_f^{r-1}h(\bar{x})]^T$, and the first subsystem of the system of Eq. B18 is the forced zero dynamics. Thus, the closed-loop system of Eq. B17 can be rewritten as

$$\begin{cases} \dot{\xi} = A_c^*\xi + b_c^*[y_{sp} - d] \\ \dot{\Theta} = b_c^* c_c^*\xi + A_c\Theta + b_c^*[y_{sp} - d] \\ \dot{\delta} = \phi(\delta, \Theta) \end{cases} \tag{B19}$$

whose Jacobian matrix is lower block-triangular. Since all the eigenvalues of the matrices A_c and A_c^* lie in the left half of the complex plane and the delay-free part of the process is minimum-phase (conditions c1, c3 and c4 hold), the closed-loop system of Eq. B19 is asymptotically stable.

Part (b): As in the proof of Part (b) of Theorem 3, by placing a root of the equation $\beta_r s^r + \cdots + \beta_1 s + 1 = 0$ sufficiently close to the origin, the controller eliminates the mismatch between y and y^* so slowly that the plant input may saturate only for an infinitesimal period of time. After this very short saturation period, the closed-loop dynamics will be governed by Eq. B16 which in Part (a) of this Theorem was proved to be asymptotically stable when the conditions c1, c2, c3 and c4 hold.

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